MMAT5120 - Topics in Geometry

Solutions to HW1

1. We are given
$$x + iy = S(a,b,c) = \frac{a+ib}{1-c}.$$
Solve
$$\begin{cases} \dfrac{a}{1-c} &= x \\ \dfrac{b}{1-c} &= y \\ a^2+b^2+c^2 &= 1 \end{cases}.$$
We have
$$|z|^2 = x^2+y^2 = \frac{a^2+b^2}{(1-c)^2} = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c}.$$
It follows that
$$c = \frac{|z|^2-1}{|z|^2+1}$$
and
$$\begin{aligned} a &= x(1-c) = \frac{2x}{|z|^2+1} \\ b &= y(1-c) = \frac{2y}{|z|^2+1}. \end{aligned}$$

2. (a) The general form of a straight line ℓ is given by
$$Ax + By + C = 0$$
where A and B are real numbers, not both equal to 0.

Then
$$f(\ell) = \begin{cases} \infty & \text{if } B = 0 \\ -\dfrac{A}{B} & \text{otherwise} \end{cases}. \tag{1}$$
Let $T : (x,y) \mapsto (x+x_0, y+y_0)$ be a translation ($x_0, y_0 \in \mathbb{R}$). Then $T(\ell)$ consists of points (x', y') satisfying
$$x' = x + x_0,\ y' = y + y_0 \quad \text{and} \quad Ax + By + C = 0.$$
It is equivalent to
$$\begin{aligned} & A(x'-x_0) + B(y'-y_0) + C = 0 \\ \iff\ & Ax' + By' + (-Ax_0 - By_0 + C) = 0. \end{aligned}$$
The last expression represents a straight line.

Let $R : x + iy \mapsto e^{i\theta}(x + iy)$ be a rotation ($\theta \in [0, 2\pi]$). Then $S(\ell)$ consists of points (x', y') satisfying

$$x' + iy' = e^{i\theta}(x + iy) \quad \text{and} \quad Ax + By + C = 0. \tag{2}$$

Notice $x + iy = e^{-i\theta}(x' + iy') = (x'\cos\theta + y'\sin\theta) + i(y'\cos\theta - x'\sin\theta)$. (2) is equivalent to

$$\begin{aligned} & A(x'\cos\theta + y'\sin\theta) + B(y'\cos\theta - x'\sin\theta) + C = 0 \\ \iff & (A\cos\theta - B\sin\theta)x' + (A\sin\theta + B\cos\theta)y' + C = 0. \end{aligned}$$

The last expression represents a straight line.

Since any transformations in the Euclidean geometry are compositions of a rotation and a translation which, as we have just seen, preserve D, the same is true for these transformations.

(b) Yes. By (a), $T(\ell)$ is represented by

$$Ax' + By' + (-Ax_0 - By_0 + C) = 0.$$

Then by (1),

$$f(T(\ell)) = -\frac{A}{B} = f(\ell).$$

This proves that f is invariant under translational geometry.

(c) No. For example, take $\ell = x-$axis. Then $f(\ell) = 0$. If we apply the rotation R by 90° clockwise to ℓ, we get $R(\ell) = y-$axis with $f(R(\ell)) = \infty \neq 0$.

3. (a)

$$(z_0, \infty; z_2, z_3) = \lim_{z \to \infty} \frac{z_0 - z_2}{z - z_2} \cdot \frac{z - z_3}{z_0 - z_3} = \frac{z_0 - z_2}{z_0 - z_3}.$$

(b) Recall the formula

$$\frac{z - z_2}{z_1 - z_2} \cdot \frac{z_1 - z_3}{z - z_3} = \frac{w - w_2}{w_1 - w_2} \cdot \frac{w_1 - w_3}{w - w_3}.$$

Put $(z_1, z_2, z_3) = (0, i, 2)$ and $(w_1, w_2, w_3) = (-2i, 1, 0)$. We get

$$w = \frac{z - 2}{2(1 + i)z - i}.$$

(c) Let $a \in \mathbb{C} - \{1, i\}$ be a variable which is sent to ∞. We look for $T_a \in \mathbb{M}$ satisfying

$$\begin{array}{ccc} 1 & \mapsto & 1 \\ i & \mapsto & i \\ a & \mapsto & \infty \end{array}.$$

By the formula above, we have ($w = T_a(z)$)

$$\frac{z - i}{z - a} \cdot \frac{1 - a}{1 - i} = \frac{w - i}{1 - i}$$

so

$$T_a(z) = \frac{(1-a+i)z - i}{z-a}.$$

Remark. There are indeed more than one expression for the answer, but each expression is related to the other by a change of variable for a.

4. Let C_1 and C_2 be two clines. Pick three distinct points $z_1, z_2, z_3 \in C_1$ and three distinct points $w_1, w_2, w_3 \in C_2$. By the fundamental theorem of Moebius geometry, there exists a (unique) $T \in \mathbb{M}$ such that $T(z_i) = w_i,\ i = 1, 2, 3$. Since Moebius transformations map clines to clines, it follows that $T(C_1)$ is a cline passing through w_1, w_2, w_3.

   Finally, using the fact that every cline is uniquely determined by any of its three distinct points, we conclude $T(C_1) = C_2$.

5. Recall z and z^* are symmetric with respect to a cline C if

$$(z, z_1; z_2, z_3) = \overline{(z^*, z_1; z_2, z_3)} \tag{3}$$

   where z_1, z_2, z_3 are any three distinct points of C.

   Recall also that Moebius transformations preserve symmetry, i.e. if z and z^* are symmetric with respect to C, then $T(z)$ and $T(z^*)$ are symmetric with respect to $T(C)$.

   (a) Suppose first that C is the $x-$axis. Choose $(z_1, z_2, z_3) = (1, 0, \infty)$. Then (3) is equivalent to

$$z = \overline{z^*}$$

   or

$$z^* = \overline{z}$$

   so symmetry is indeed equivalent to reflection in the usual sense.

   Assume now C is a general straight line. Then there exists a transformation S in the Euclidean geometry (i.e. the composite of a translation and a rotation) sending C to the $x-$axis. The key point is that S preserves both symmetry (being a Moebius transformation) and reflection (being a rigid motion). It follows that if

$$z \text{ and } z^* \text{ are symmetric with respect to } C,$$

   then

$$S(z) \text{ and } S(z^*) \text{ are symmetric with respect to the } x\text{-axis}.$$

   From what we have proved above, we have

$$S(z) \text{ is the reflection of } S(z^*) \text{ with respect to the } x\text{-axis},$$

   and hence

$$z = S^{-1}(S(z)) \text{ is the reflection of } S^{-1}(S(z^*)) = z^* \text{ w.r.t } S^{-1}(x\text{-axis}) = C.$$

(b) In this part and the next, we may assume C and C' are general clines.

Choose a $T \in \mathbb{M}$ sending an intersection point of C and C' to ∞. Then $T(C)$ and $T(C')$ are two perpendicular straight lines (as T preserves orthogonality). Moreover $T(z)$ and $T(z^*)$ are symmetric with respect to $T(C)$.

By (a), $T(z)$ is the reflection of $T(z^*)$ with respect to $T(C)$. As $T(z) \in T(C')$ and $T(C') \perp T(C)$, we see that $T(C')$ also passes through $T(z^*)$, and hence C' passes through z^*.

(c) First we show that C and C' must intersect. Suppose not, choose a $T \in \mathbb{M}$ sending C to the $x-$axis. Then $T(C')$ is a circle not intersecting the $x-$axis. (It cannot be a straight line, otherwise C' intersects C.) So $T(C')$ lies either in the upper half-plane or the lower half-plane.

But according to (a), any pair of symmetric points are reflection of each other. It must be that exactly one of them lies in the upper half-plane and exactly one lying in the lower half-plane. It follows that $T(C')$ cannot pass through both of these points, a contradiction.

Now it is clear that, no matter C and C' are circles or straight lines, there is always a $T \in \mathbb{M}$ sending them to a pair of intersecting straight lines $T(C)$ and $T(C')$ whose intersection point ($\neq \infty$) corresponds to any given intersection point of C and C'. We need to show that $T(C)$ and $T(C')$ are perpendicular. But as argued above, this follows from the assumption that $T(C')$ contains a pair of reflection points with respect to $T(C)$.